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Analyticity Properties and a Convergent Expansion for the Inverse Correlation Length of the Low-Temperature d-Dimensional Ising Model

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We show that the inverse correlation length $m(z)$ of the truncated spin–spin correlation function of the Z^d Ising model with + or − boundary conditions admits the representation $m(z) = -(4d-4)\ln z(1-\delta_{d1}) + r(z)$ for small $z = e^{-\beta}$, i.e., large inverse temperatures $\beta > 0$. $r(z) = \sum_{n=1}^{\infty} b_n z^n$ is a d-dependent analytic function at $z = 0$, already known in closed form for $d = 1$ and 2; for $d \geqq 3$ b_n can be computed explicitly from a finite number of the Z^d limits of $z = 0$ Taylor series coefficients of the finite lattice correlation function at a finite number of points of Z^d.

KEY WORDS: Ising model; correlation length; correlation length expansion; low-temperature Ising model; correlation function.

In this short note analytic properties and a convergent expansion are obtained for the inverse correlation length of the truncated spin–spin correlation function (cf) of the nearest-neighbor spin ± 1 Z^d Ising model with $\pm$ boundary conditions for large inverse temperature β. Our results are analogous to those obtained in Ref. 1 for the high-temperature Ising model and follow easily upon combining the techniques of Ref. 1 and the results of Ref. 2. For $d = 1$ and 2 our results are well known from explicit formulas (see Ref. 3) and so will not be discussed further.

We denote by $G_\Lambda(x; y, z) = \langle s_x s_y \rangle_\Lambda - \langle s_x \rangle_\Lambda \langle s_y \rangle_\Lambda$, $x, y \in \Lambda$ the truncated spin–spin cf for the finite lattice $\Lambda \subset Z^d$ with $\pm$boundary conditions

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and complex activity z where for $F(s) \equiv \prod_{i \in A} s_i$, $A \subset \Lambda$, we define

$$\langle F \rangle_\Lambda = \frac{1}{Z_\Lambda} \sum_{\{s\}} F(s) \prod_{\langle i,j \rangle \subset \Lambda} z^{(1 - s_i s_j)}$$

$\langle i, j \rangle$ denote the unordered nearest neighbor pairs of Λ and the sum is over all spin configurations $[-1, 1]^\Lambda$ with the restriction that the boundary spins are $+1$. Z_Λ is a normalization factor such that $\langle 1 \rangle_\Lambda = 1$. $z = e^{-\beta}$, $\beta > 0$, corresponds to the physical model. A similar definition holds for $-$ boundary conditions but for definiteness we consider plus conditions throughout.

We let $G(x; y, z)$ denote the Z^d lattice cf defined by

$$G(x; y, z) = \lim_{\Lambda \uparrow Z^d} G_\Lambda(x; y, z)$$

By Ref. 2 $G(x; y, z)$ exists, is translation invariant and analytic in z for $|z|$ small. By translation invariance we can write $G(x; y, z) \equiv G(x - y, z)$. We let $x = (x_1, \ldots, x_d) = (x_1, \mathbf{x})$ denote points of Z^d, $|x| = \sum_{i=1}^{d} |x_i| = |x_1| + |\mathbf{x}|$, and let $\tilde{G}(p, z) = \sum_x e^{ipx} G(x, z)$ denote the Fourier transform of $G(x, z)$ where $p = (p_1, \mathbf{p})$, $p_i \in (-\pi, \pi]$ and $px = \sum_{i=1}^{d} p_i x_i$. In Ref. 2 a lattice quantum field theory is associated with the Ising model correlation functions and, for $d \geqq 3$, $z > 0$ and small, it is shown that there is an isolated dispersion curve $\omega(\mathbf{p})$, real analytic in $\mathbf{p} \in (-\pi, \pi]^{d-1}$, $\omega(\mathbf{p}) \geqq \omega(\mathbf{0}) \equiv m(z)$ where $m(z)$, the inverse correlation length ($=$ mass of the fundamental particle of the associated quantum field theory), is defined by

$$m(z) = \lim_{x_1 \to \infty} \frac{-1}{x_1} \ln G(x = (x_1, \mathbf{0}), z)$$

$\omega(\mathbf{p})$ is defined by

$$\omega(\mathbf{p}) = \lim_{x_1 \to \infty} \frac{-1}{x_1} \ln \Big[\sum_{\mathbf{x}} e^{i\mathbf{p} \cdot \mathbf{x}} G(x, z) \Big]$$

Furthermore the mass and dispersion curve satisfy $\lim_{z \downarrow 0} [m(z) / -(4d - 4) \ln z] = 1$, $\lim_{z \downarrow 0} [\omega(\mathbf{p}) / m] = 1$, uniformly in $\mathbf{p} \in (-\pi, \pi]^{d-1}$. In Ref. 5 expansions for $m(z)$ are obtained.

We state our results as Theorems 1 and 2.

Theorem 1. (a) There exists a function $r(z)$, analytic at $z = 0$, $r(0) = 0$, such that for all $z = e^{-\beta} > 0$ and small $m(z)$ admits the representation $m(z) = -(4d - 4) \ln z + r(z)$.

(b) There exists a $z' > 0$ such that for each $z \in (0, z')$ $m(z)$ is analytic.

Remark. z' may be larger than the radius of convergence of the $z = 0$ Taylor series of $r(z)$.

Theorem 2. $b_n \equiv (1/n!)(d^n r/dz^n)$ $(z = 0)$, the nth Taylor series coefficient of $r(z)$ can be computed from a finite number of the Z^d limits of the

$z = 0$ Taylor series coefficients of the finite lattice cf $G_\Lambda(0; x, z)$ for a finite number of x, $|x| < R(n)$, where $R(n)$ increases with n.

Similar results hold for the dispersion curve $\omega(\mathbf{p})$. The proofs of Theorems 1 and 2 will be given after some preliminary lemmas.

The results of Ref. 2 as well as ours follow from z analyticity and x-decay properties of $G(x,z)$ and $\Gamma(x,z)$, where $\Gamma(x,z) = \Gamma(y; u, z)$, $x = y - u$, is the convolution inverse of $G(x; y, z)$ interpreted as a matrix operator on $l_2(Z^d)$, i.e., $\Gamma(x; y, z) = G^{-1}(x; y, z)$, $\sum_u \Gamma(x; u, z) G(u; y, z) = \delta_{xy}$. Specifically $\Gamma(x,z)$ has faster x_1 falloff than $G(x,z)$. We state the results of Ref. 2 in the form needed here as Lemmas 1 and 2. We often drop the z argument for notational simplicity. We let $\| \ \|$ denote the $l_2(Z^d)$ operator norm and $|\mathbf{x}|_\infty \equiv \sup_{2 \leqq i \leqq d} |x_i|$. In what follows all results are to be understood as holding for $d \geqq 3$ and all sufficiently small $|z|$ unless stated otherwise; $c, c', c_1, \ldots$ will denote strictly positive constants.

Lemma 1. (a) There exist c, c_1, c_2 such that $G(x,z)$ is analytic in z,

$$|G(x,z)| \leqq c_1 |c_z|^{4(d-1)|x_1| + 4|\mathbf{x}|_\infty + 4d},$$

$\|G\| < c_2$, and $G(x_1, \mathbf{x}, z) = G(-x_1, \mathbf{x}, z)$.

(b) $\tilde{G}(p,z)$ is jointly analytic in z and p_1, $|\mathrm{Im}\, p_1| < -4(d-1)\ln|cz|$.

Lemma 2. (a) Let $P: l_2(Z^d) \to l_2(Z^d)$ be the operator with matrix elements $P(x; y, z) = G(x,z)\delta_{xy}$, then P is analytic, $|P(x,z)| \geqq |z|^{4d}$ and for $|z| > 0$ P^{-1} exists, is analytic and $\|P\| \leqq |z|^{-4d}$.

(b) Let $Q = P^{-1}(G - P)$ then there exist c, c_1 such that Q is analytic,

$$|Q(x,z)| \leqq c_1 |cz|^{4(d-1)|x_1| + 4|\mathbf{x}|_\infty}(1 - \delta_{x0})$$

and $\|Q\| < 1/2$.

(c) $M \equiv (I + Q)^{-1} = \sum_{n=0}^{\infty} (-1)^n Q^n$ is analytic and the series is norm convergent. There exists c_2, c_3, c' such that $\|M\| < c_2$,

$$|M(x,z)| \leqq c_3 |c'z|^{(4d-3)|x_1| - 1 + 4|\mathbf{x}|_\infty}$$

for $|x_1| \geqq 1$; for $x = (0, \mathbf{x})$, $|M(x,z)| \leqq c_3 |c'Z|^{4|\mathbf{x}|_\infty}$. $\tilde{M}(p,z)$ is jointly analytic in z and p_1 in $|\mathrm{Im}\, p_1| < -(4d-3)\ln|c'z|$.

(d) For $|z| > 0$, $\Gamma = MP^{-1}$ and is analytic. There exist c_3, c_4, c_5, c' such that

$$|\Gamma(x,z)| \leqq c_4 |c_5 z|^{(4d-3)|x_1| - (4d+1) + 4|\mathbf{x}|_\infty}$$

for $|x_1| \geqq 1$; for $x = (0, \mathbf{x})$, $|\Gamma(x,z)| \leqq c_3 |c'z|^{4|\mathbf{x}|_\infty - 4d}$.

(e) For $|z| > 0$, $\tilde{\Gamma}(p,z) = \tilde{M}(p,z) G(0,z)^{-1}$ is jointly analytic in z and p_1 in $|\mathrm{Im}\, p_1| < (4d-3)\ln|c_5 z|$.

(f) For $z \neq 0$, $\tilde{\Gamma}(p,z)\tilde{G}(p,z) = 1$ in the analyticity region of $\tilde{G}(p,z)$.

(g) For $z > 0$, $p_1 = i\omega(\mathbf{p})$ satisfies $\tilde{\Gamma}(p_1 = i\omega(\mathbf{p}), \mathbf{p}, z) = 0$ and is the only zero of $\tilde{\Gamma}(p,z)$ in $0 < \operatorname{Im} p_1 < -(4d-3)\ln|c_5 z|$, $|\operatorname{Re} p_1| < \pi$, is simple and $(\partial\tilde{\Gamma}/\partial p_1)(p_1 = i\omega(\mathbf{p}), \mathbf{p}, z) = Z'(\mathbf{p}) > 0$.

Remark. The bound on $Q(x,z)$ of (b) is obtained by including the $\mathbf{x}$ decay from Theorem A1.1 of Appendix I in the arguments of Section 3 of Ref. 2. The crucial bound on $M(x,z)$ of (c) follows by including $\mathbf{x}$ decay and combining Eq. (3.7) and Theorem 3.4. of Ref. 2.

The proofs of Theorems 1 and 2 are based on the implicit equation for $m(z)$ of Lemma 2g as the zero of $\tilde{\Gamma}$. However, Γ and $\tilde{\Gamma}$ are not analytic at $z = 0$ but $\Gamma P = M$ and $\tilde{M}$ are and as $|P(0,z)| \geqq |z|^{4d}$ by Lemma 2a the zero at $p = (\operatorname{im}(z), 0)$ of $\tilde{M}(p,z) = P(0,z)\tilde{\Gamma}(p,z)$ is the zero of $\tilde{\Gamma}(p,z)$. Thus we look for the zero of $\tilde{M}$. By Lemma 2b, c we are led to write the $z = 0$ Taylor expansion for $\tilde{M}(p,z)$ with the terms up to and including order $z^{4(d-1)}$ explicit. The explicit terms of $\tilde{M}(p,z)$ are obtained from the $z = 0$ Taylor series of $\tilde{G}(p,z), G(0,z)$ (see Lemma 3 below) and the relation $\tilde{M}(p,z) = \tilde{\Gamma}(p,z)G(0,z) = \tilde{G}(p,z)^{-1}G(0,z)$ using Lemma 2e, f. Let

$$M_s(x,z) \equiv M(x,z) - \sum_{m=0}^{4d-4} \frac{z^m}{m!} \frac{\partial^m M}{\partial z^m}(x, z=0)$$

$$= \frac{z^{4d-3}}{(4d-4)!} \int_0^1 (1-t)^{4d-4} \frac{\partial^{4d-3}}{\partial \xi^{4d-3}} M(x, \xi = zt)\, dt$$

$$\tilde{M}_s(p_1, \mathbf{p} = 0, z) \equiv \tilde{M}_s(p_1, z) \quad \text{and for} \quad n = 0, 1, \ldots$$

$$M_s(n,z) = \sum_{\mathbf{x}} M_s(x_1 = n, \mathbf{x}, z)$$

Lemma 3. For $|\operatorname{Im} p_1| < -4(d-1)\ln|cz|$ $\tilde{G}(p,z)$ has the $z = 0$ Taylor expansion

$$\tilde{G}(p,z) = 4z^{4d} + 8\,dz^{8d-4} + 4z^{8d-4}(e^{-ip_1} + e^{ip_1})$$
$$+ 8z^{8d-4} \sum_{i=2}^{d} \cos p_i + O(z^{8d-3})$$

the x series of $\tilde{G}(p,z)$ converges absolutely.

Proof. By a consideration of Pierels contours in the expansion of Ref. 2 the only x that contribute to $\tilde{G}(p,z)$ up to order z^{8d-4} are $x = 0$ and x, $|x| = 1$. The $z = 0$ expansions of $G(x = 0,\ z)$ and $G(x,z)$, $|x| = 1$, are carried out using the duplicate variable representation as in Ref. 2, the first two terms coming from $G(0,z)$. ■

Lemma 4. (a) $|M_s(x,z)| \leqq c_2|cz|^{4d-3}$.

(b) $\tilde{M}(p,z)$ is jointly analytic in z and p_1 in $|\mathrm{Im}\, p_1| < -(4d-3)\ln|cz|$ and has the $z=0$ Taylor expansion

$$\tilde{M}(p,z) = 1 - z^{4d-4}(e^{-ip_1} + e^{ip_1}) - 2z^{4d-4}\sum_{i=2}^{d}\cos p_i + \tilde{M}_s(p,z)$$

the x series of $\tilde{M}$ and $\tilde{M}_s$ converge absolutely and there exists c_6 such that $|\tilde{M}_s(p,z)| \leqq c_6|z|^{4d-3}$.

(c) $M_s(n,z)/z^{(4d-4)n}$ is analytic, the **x** series converges absolutely, and there exists c_7 such that $|M_s(n,z)| \leqq c_7|c'z|^{(4d-3)n}$, $n \neq 0$; $|M_s(n=0,\ z)| \leqq c|z|^{4d-3}$.

Proof. The proof of (a) follows by a Cauchy estimate on $(\partial^{4d-3}/\partial\xi^{4d-3})M(x,\xi = zt)$ using Lemma 2c. For (b) the analyticity follows using the bounds of Lemma 2c and the explicit terms are obtained from $\tilde{M}(p,z) = \tilde{\Gamma}(p,z)G(0,z) = \tilde{G}(p,z)^{-1}G(0,z)$. The proof of (c) follows using (a) and noting that $M(x,z) = M_s(x,z)$ for x such that $(4d-3)|x_1| - 1 + 4|\mathbf{x}|_\infty \geqq 4d-3$. ■

We now give the proofs of the theorems. For the proof of Theorem 1b we refer to Ref. 1.

Proof of Theorem 1a. $\tilde{M}(p_1,z)$ can be written

$$\tilde{M}(p_1,z) = 1 - 2z^{4d-4}(d-1) - z^{4d-4}(e^{-ip_1} + e^{ip_1}) + M_s(n=0,\ z) + \sum_{n=1}^{\infty} M_s(n,z)(e^{-ip_1 n} + e^{ip_1 n})$$

Introduce the auxiliary complex variable w and function $H(w,z)$ such that $H(w = z^{4d-4}e^{-ip_1} - 1,\ z) = \tilde{M}(p_1,z)$ where

$$H(w,z) = w - \frac{z^{8d-8}}{1+w} - 2z^{4d-4}(d-1) + M_s(n=0,\ z) + \sum_{n=1}^{\infty} M_s(n,z)\left[\frac{(1+w)^n}{z^{n(4d-4)}} + \frac{z^{n(4d-4)}}{(1+w)^n}\right]$$

Using the estimates of Lemma 4c and the ratio test we find that $H(w,z)$ is jointly analytic in w and z for $|w|, |z|$ small, $H(0,0)=0$ and $(\partial H/\partial w)(0,0) = 1$. Thus by the analytic implicit function theorem there exists a unique analytic function $w(z)$, $w(0)=0$, such that $H(w(z),z)=0$. For $z>0$ $w(z) = z^{4d-4}e^{m(z)} - 1$ or $m(z) = -(4d-4)\ln z + r(z)$ with $r(z) = \ln(1+w(z))$. ■

Proof of Theorem 2. The argument is as in Ref. 1 so we only give a sketch here. The $z=0$ Taylor series coefficients of $r(z)$ are determined from those of $w(z)$ which depend on those of M. The $z=0$ Taylor coefficients of $M(x,z)$ are determined from a finite number of those of $Q=P^{-1}(G-P)$ for a finite number of points x of $Q(x,z)$. Here we have used the falloff of M and Q given by Lemma 2b, c. As $Q(x,z)=G(0,z)^{-1}[G(x,z)-G(0,z)]$, $x\neq 0$, the $z=0$ Taylor series coefficients are determined from those of $G(0,z)$ and $G(x,z)$.

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